Comparison of Bit and Word Level Algorithms for Evaluating Unstructured Functions over Finite Rings

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Motivation

Need to implement unstructured functions defined over finite fields or rings:

- S-boxes in block and stream ciphers (DES, AES)
- Round functions in hash functions (MD5, SHA-1)
- Public key schemes defined over finite fields or rings

Implementation

- Common representation

$$f(x_1, x_2, x_3) = c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_1x_2 + c_5x_1x_3 + c_6x_2x_3 + c_7x_1x_2x_3$$

  where $c_i, x_i \in R$.
- Typically implemented as parallel circuit *as given in the description*
- Components of the circuit are isolated blocks implementing operations in R.

A Question

- Idea: We can view the entire function as being defined over $GF(2)$
- Which approach is more efficient?
  - implement the circuit in two levels first as a circuit over R, and then implement operations in R as boolean circuits
  - implement the whole circuit as a boolean circuit, i.e. over $GF(2)$.

Horner's Method

In the univariate case a polynomial of degree $r-1$ over Z_m is represented as

$$u(x) = u_0 + u_1 x + u_2 x^2 + \ldots + u_{r-1} x^{r-1} \quad , \quad u_i \in Z_m \ .$$

Applying Horner's method

$$u(x) = u_0 + x(u_1 + x(u_2 + x(u_3 + \ldots + x(u_{r-2} + x u_{r-1})) \ldots)$$

is evaluated by computing only $r-1$ additions and $r-1$ multiplications with delay $T = (r-1)T_A + (r-1)T_M$

The Multivariate Version of Horner's Method

Level	#Coefficient Polynomials	#Mult or #Add
1	r	$(r-1)$
2	r^2	$(r-1)r$
3	r^3	$(r-1)r^2$
$\vdots$	$\vdots$	$\vdots$
n	r^n	$(r-1)r^{n-1}$

Table: Number of coefficient polynomials introduced in each level

The Multivariate Version of Horner's Method

- The evaluation of an n-variate polynomial over Z_m of maximum degree $(r-1)$ in all variables requires at most $r^n - 1$ additions and $r^n - 1$ multiplications in Z_m.
- The delay of a parallel circuit (of n levels) is at most $T = n(r-1)T_A + n(r-1)T_M$.

An Example

Let $Z_m = Z_2$ and $f = f(x_1, x_2, x_3, x_4)$ represent a multivariate polynomial $f : (Z_2)^4 \mapsto Z_2$ explicitly given as

$$\begin{aligned} f &= x_1x_2x_3x_4 + x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4 + x_1x_3 + x_3x_4 \\ &+ x_2x_4 + x_3x_4 + x_3 + x_2 + x_1 + 1 \ . \end{aligned}$$

Applying Horner's algorithm we convert the polynomial into the following representation

$$\begin{aligned} f &= 1x_1\left[1x_2\{1x_3(1x_4+1)+(1x_4+0)\}+\{1x_3(0x_4+1)+(1x_4+1)\}\right] \\ &+ \left[1x_2\{1x_3(1x_4+0)+(1x_4+1)\}+\{1x_3(1x_4+1)+(0x_4+1)\}\right] \end{aligned}$$

An Observation

- In the last level we have 8 polynomial evaluations of the form $ax_4 + b$ where $a, b \in Z_2$.
- However, there can be only 2^2 such polynomials.
- Multivariate version of Horner's algorithm is redundant!
- Same argument can be repeated for lower levels as well.
- Need to find the level where redundancy vanishes.

The Optimization Strategy

Level	#Coefficient Polynomials	#Mult or #Add	#Unique Polynomials	#Mult or #Add
1	r	$(r-1)$	m^{nr}	$(r-1)m^{r^n}$
2	r^2	$(r-1)r$	$m^{(n-1)r}$	$(r-1)m^{r^{n-1}}$
3	r^3	$(r-1)r^2$	$m^{(n-2)r}$	$(r-1)m^{r^{n-2}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	r^{n-2}	$(r-1)r^{n-3}$	m^{3r}	$(r-1)m^{r^3}$
$n-1$	r^{n-1}	$(r-1)r^{n-2}$	m^{2r}	$(r-1)m^{r^2}$
n	r^n	$(r-1)r^{n-1}$	m^r	$(r-1)m^r$

Table: Number of coefficient polnomials and unique polynomials at each level

Finding the Sweetspot

- Find the level k in which the number of coefficients exceeds the number of unique polynomials
- Find the smallest value of k satisfying

$$r^k \geq m^{r^{n-k+1}}$$

- Take the logarithm of both sides

$$kr^k \geq r^{n+1} \log_r m \ .$$

- Define $c = r^{n+1} \log_r m$ and take the log of both sides w.r.t base r

$$k \geq \log_r c - \log_r k \ .$$

Finding the Sweetspot

- Keep substituting value of k

$$k = \log_r c - \log_r(\log_r c - \log_r k(\log_r c - \log_r k(\log_r c - \log_r(\ldots)\ldots)\ldots) .$$

- The exact solution is defined in terms of the Lambert-W function [2]

$$k \geq W(\log r \frac{r^{n+1}}{\log_m r}) / \log r$$

where $W(x)$ is defined as the inverse of the map $x \rightarrow xe^x$.

- Approximate k by neglecting terms after two levels of substitution

$$k \approx \log_r c - \log_r(\log_r c) .$$

The Circuit Complexity

- Derive complexity in terms of Z_m additions and multiplications

$$
\begin{aligned}
C &= \sum_{i=1}^{k}(r-1)r^{i-1} + \sum_{i=1}^{n-k}(r-1)m^{r^i} \\
&= (r^k - 1) + (r-1)(m^r + m^{r^2} + m^{r^3} + \ldots + m^{r^{n-k}}) \\
&\approx r^k + rm^{r^{n-k}}
\end{aligned}
$$

- Substitute values derived from other identities[1]

$$
\begin{aligned}
C &= \frac{c}{\log_r c} + rm^{\frac{n \log_m r}{r}} \\
&= \frac{r^{n+1}\log_r m}{(n+1) + \log_r(\log_r m)} + r^{\frac{n}{r}+1}
\end{aligned}
$$

- Addition and multiplication complexities grow by $O(\frac{r^n}{n})$.

[1] See paper for details

Modified Horner over Prime Fields $GF(p)$

- Given $n > p$ the evaluation of an n-variate polynomial over $GF(p)$ requires at most $O(\frac{p^n}{n})$ additions and multiplications in $GF(p)$ with a delay of $O((p-1)(n-\log_p n))$.
- Muller [5] gives a construction gives a method for evaluating arbitrary n-variate polynomials over $GF(2)$ with $O(\frac{2^{n+1}}{n+1})$ complexity
- For $p = 2$ our construction is equivalent to Muller's construction.

Comparison of Circuit Area

- The bit-level algorithm implementing a polynomial evaluation over $GF(p)$ has bit-complexity

$$C_B = O\left((\log_2 p)\frac{2^{n\log_2 p+1}}{n\log_2 p + 1}\right) = O\left(\frac{2p^n}{n}\right) .$$

- Assuming a $GF(p)$ multiplication operation takes $(\log_2 p)^2$ bit operations we obtain the bit complexity of word level evaluation as follows

$$C_W = O\left(\frac{p^{n+1}}{n+1}(\log_2 p)^2\right) .$$

- The bit-level algorithm is $\frac{p}{2}(\log_2 p)^2$ times more area efficient

Comparison of Time Complexities

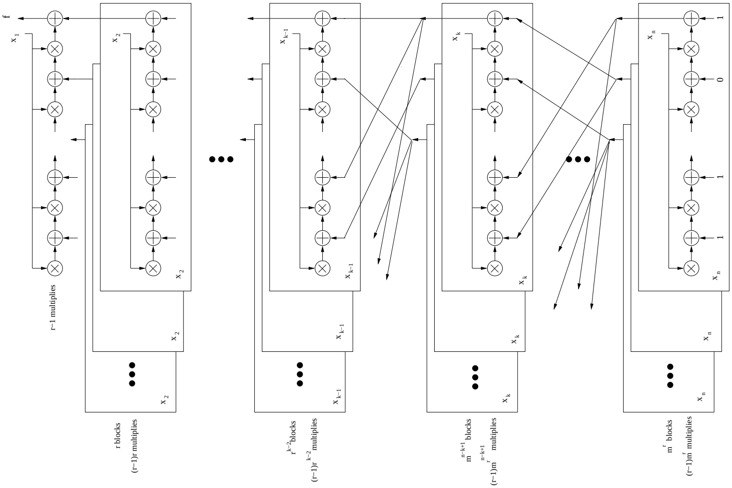
- The bit-level approach yields a time complexity of

$$T_B = O(n \log_2 p - \log_2(n \log_2 p)).$$

- Ignoring the constant operations the overall computation takes

$$T_W = O((p-1)(\log_2 \log_2 p)(n - \log_p n)).$$

  gate delays in the word-level approach.
- The bit-level algorithm is roughly $\frac{(p-1)(\log_2 \log_2 p)}{\log_2 p}$ times faster


f
r−1 multiplies
r blocks
(r−1)r multiplies
r^{k−2} blocks
(r−1)r^{k−2} multiplies
m^{n−k+1} blocks
(r−1)m^{n−k+1} multiplies
m^r blocks
(r−1)m^r multiplies
0
1

Conclusion

- We have develop a generic technique for *optimally* implements multivariate functions defined over finite rings.
- We have shown that implementing arbitrary (or generic) circuits over $GF(2)$ is more efficient
- The bit-level algorithm is $\frac{p}{2}(\log_2 p)^2$ times more area efficient
- The bit-level algorithm is roughly $\frac{(p-1)(\log_2 \log_2 p)}{\log_2 p}$ times faster
- Fan-out may be a problem for the bit-level algorithm!

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