Constructing S-boxes for lightweight cryptography with Feistel Structure

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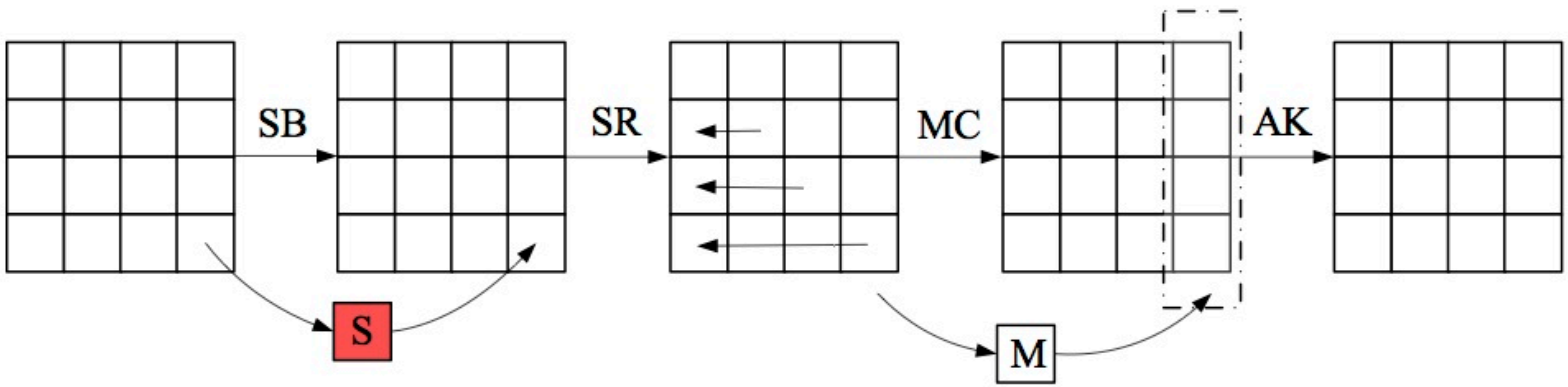
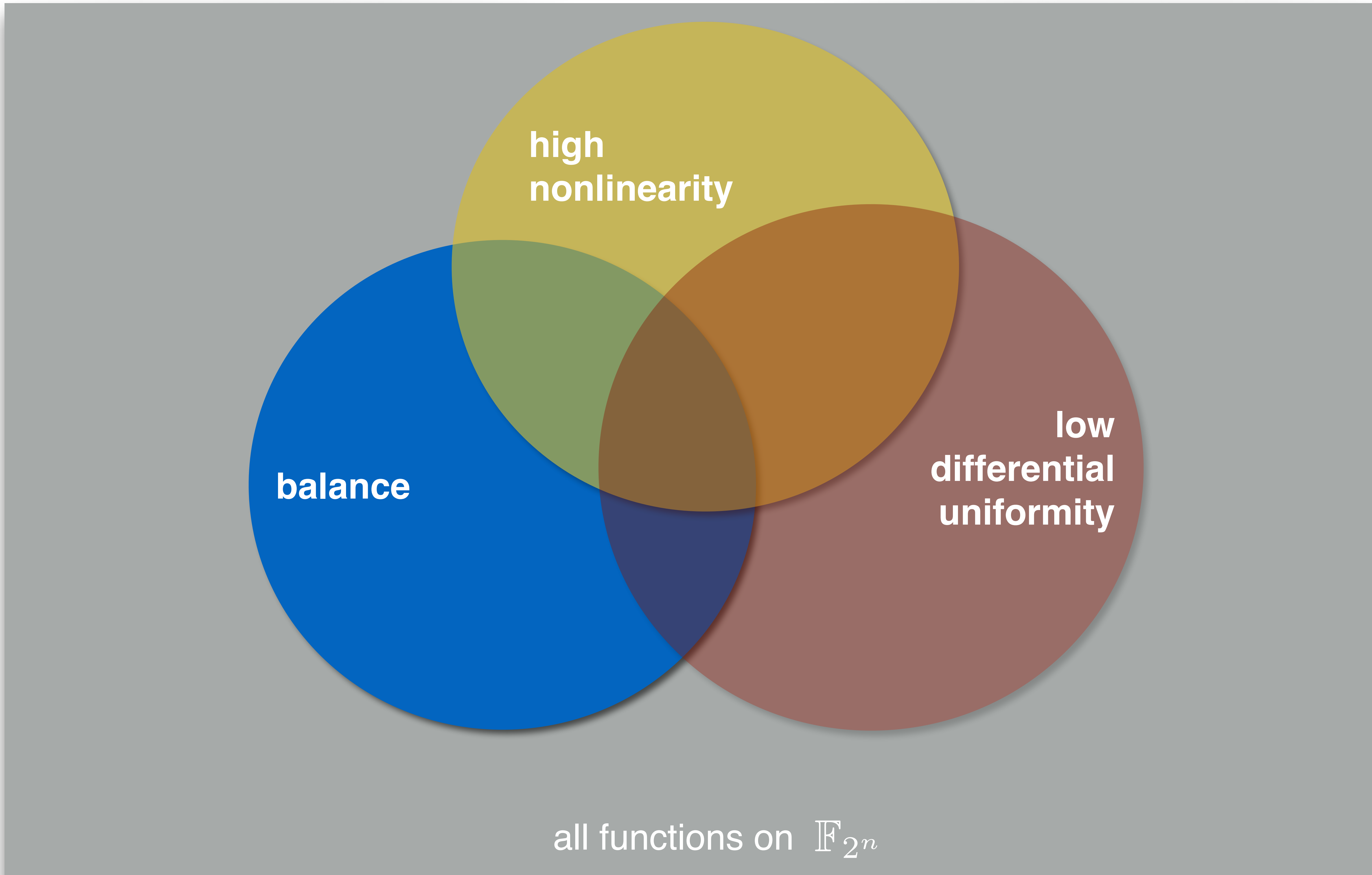
The role of S-boxes in symmetric cryptography



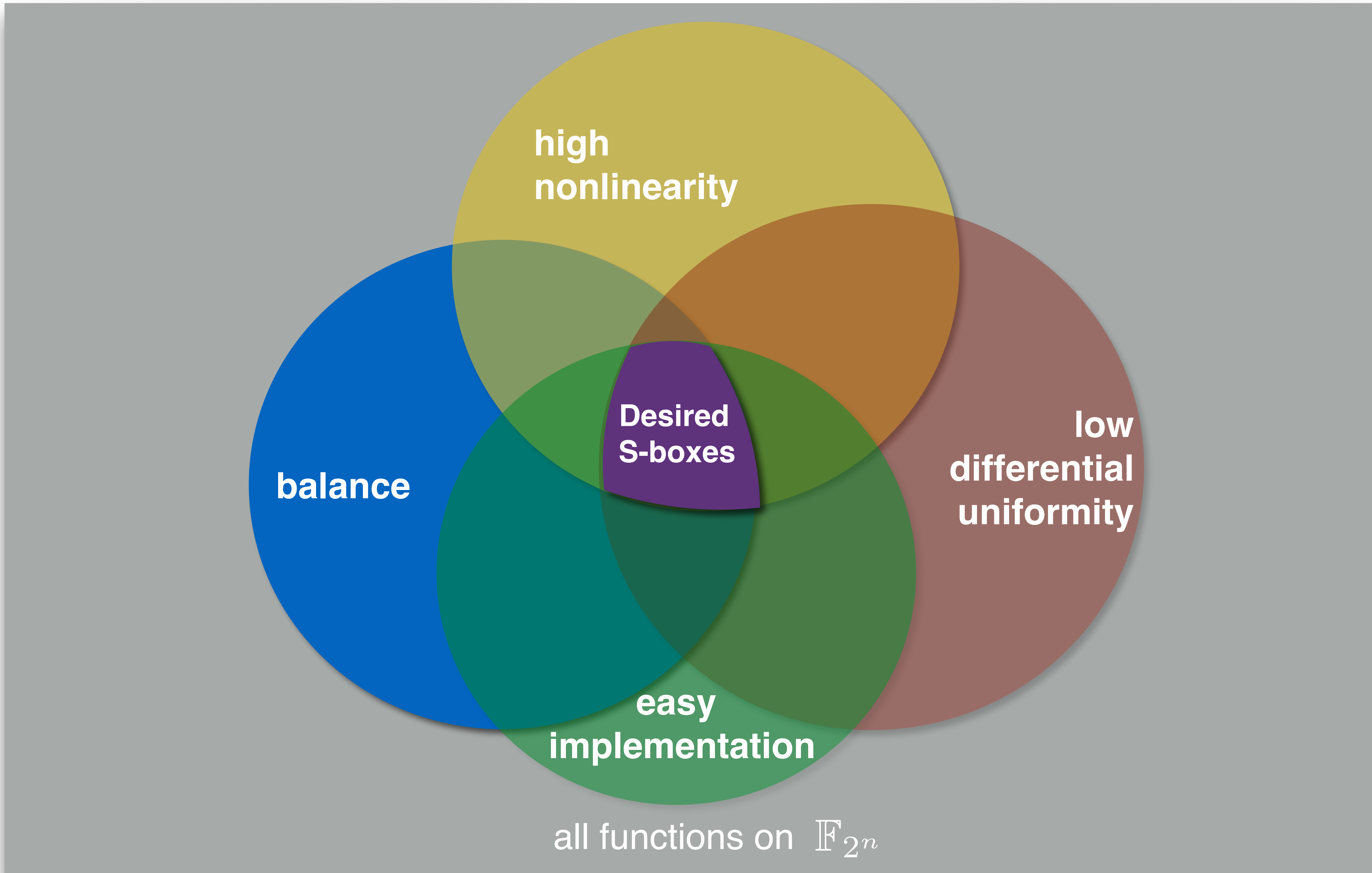
- Provide ``confusion";
- Only nonlinear part of round functions for most algorithms.

Remark: All S-boxes in this talk are n-bit S-boxes.

Basic cryptographic properties of S-boxes



S-boxes for lightweight cryptography

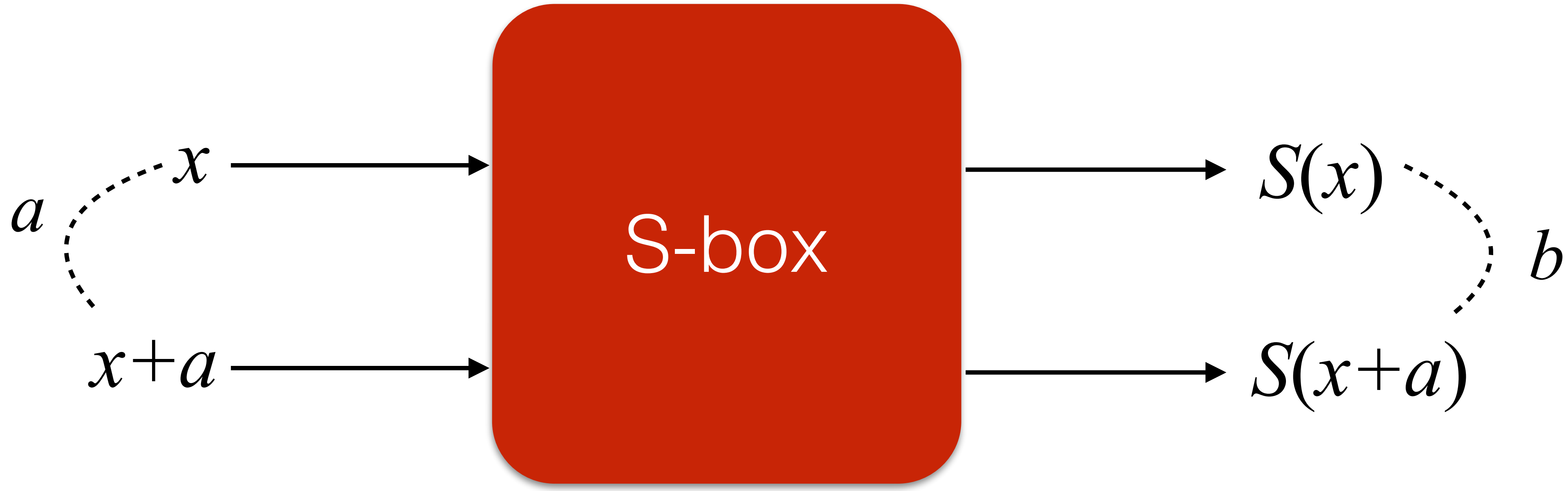


Differential uniformity

$$\Delta(S) = \max_{a \in \mathbb{F}_{2^n}^*, b \in \mathbb{F}_{2^n}} |\{x \in \mathbb{F}_{2^n} : S(x) + S(x+a) = b\}|$$

$$\Delta(S) \geq 2$$

Functions with equality holds are called almost perfect nonlinear (APN) functions.



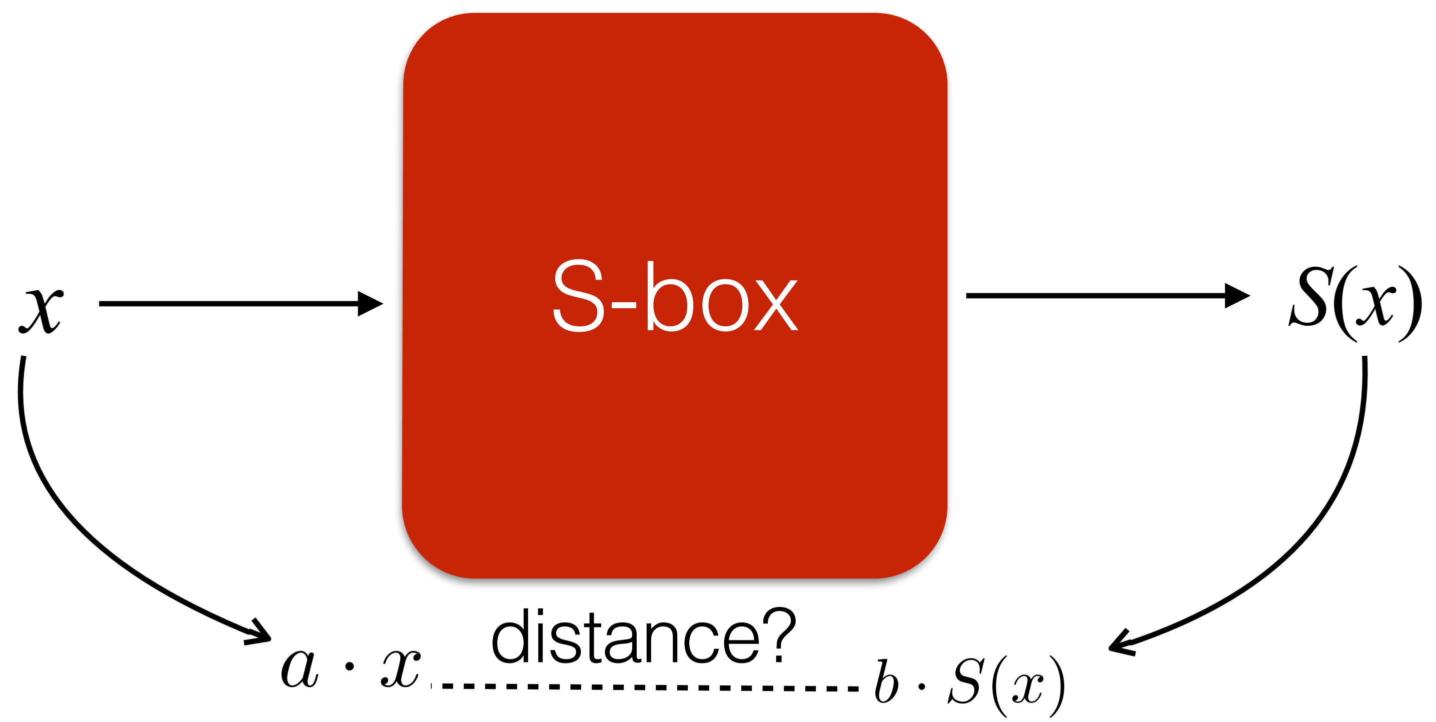
S-boxes with lower differential uniformity posses better resistance to differential attack.

Nonlinearity

The minimal distance of all the components of $S(x)$ to affine Boolean functions.

$$\lambda_S(a,b) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{\mathrm{Tr}(bS(x)+ax)}$$

$$\mathcal{NL}(S) = 2^{n-1} - \frac{1}{2}\max\{|\lambda_S(a,b)| : a, b \in \mathbb{F}_{2^n}, b \neq 0\}$$



S-boxes with higher nonlinearity posses better resistance to linear attack.

The best performance of nonlinearity and differential uniformity of permutations over $\mathbb{F}_{2^n}$

nonlinearity

$$\mathcal{NL}(S) \leq \begin{cases} 2^{n-1} - 2^{\frac{n-1}{2}} & n \text{ odd} \\ 2^{n-1} - 2^{\frac{n}{2}} & n \text{ even} \end{cases}$$

differential uniformity

$$\Delta(S) \geq \begin{cases} 2 & n \text{ odd} \\ 4 & n = 4 \\ 2 & n = 6 \\ 4 & n \geq 8 \text{ even} \end{cases}$$

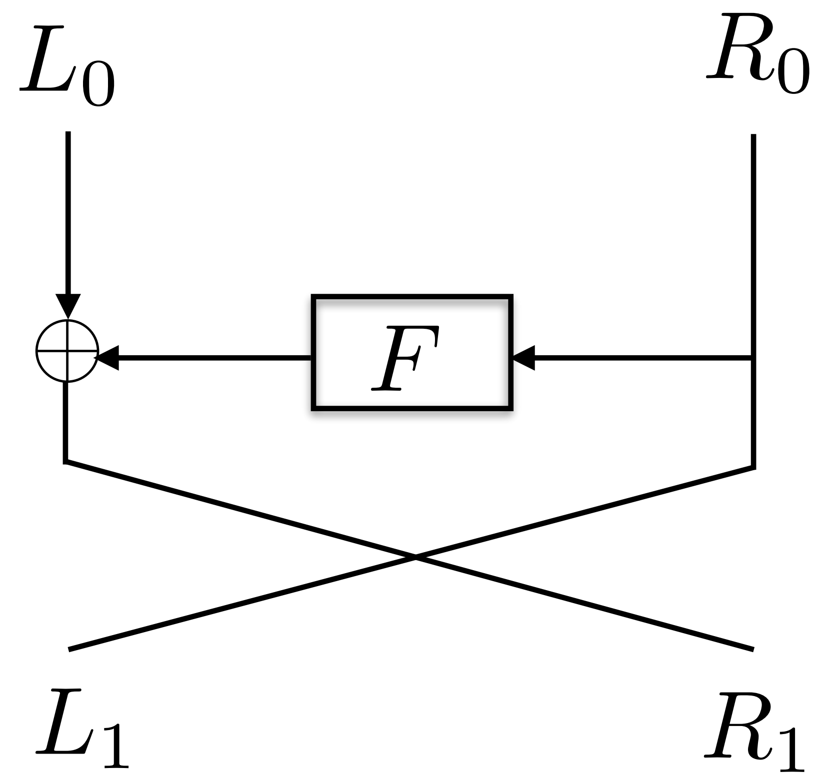
The two red bounds above are not proven yet.

The main problem

Construct S-boxes with the following properties:

- n even, permutation;
- Lowest differential uniformity; 4
- The best known nonlinearity; $2^{n-1} - 2^{\frac{n}{2}}$
- Easy implementation;

Feistel structure

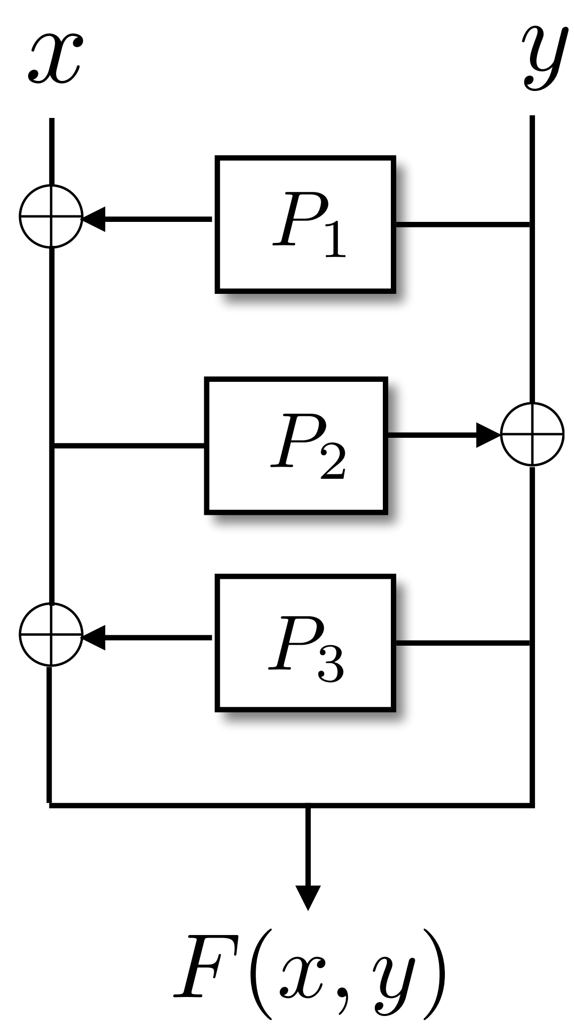


$$(L_0, R_0) \to (R_0, L_0 + F(R_0))$$

Feistel structure has low implementation cost.

S-boxes constructed with 3-round Feistel structure

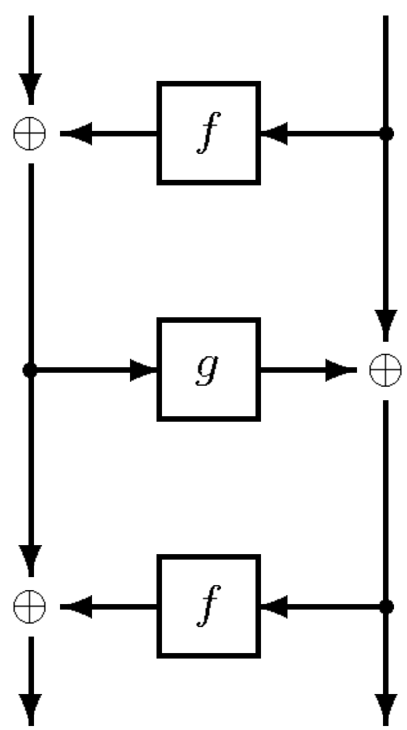
$$x, y \in \mathbb{F}_{2^k}, P_1, P_2, P_3 \in \mathbb{F}_{2^k}[x].\ F(x, y) : \mathbb{F}_{2^k}^2 \longrightarrow \mathbb{F}_{2^k}^2$$



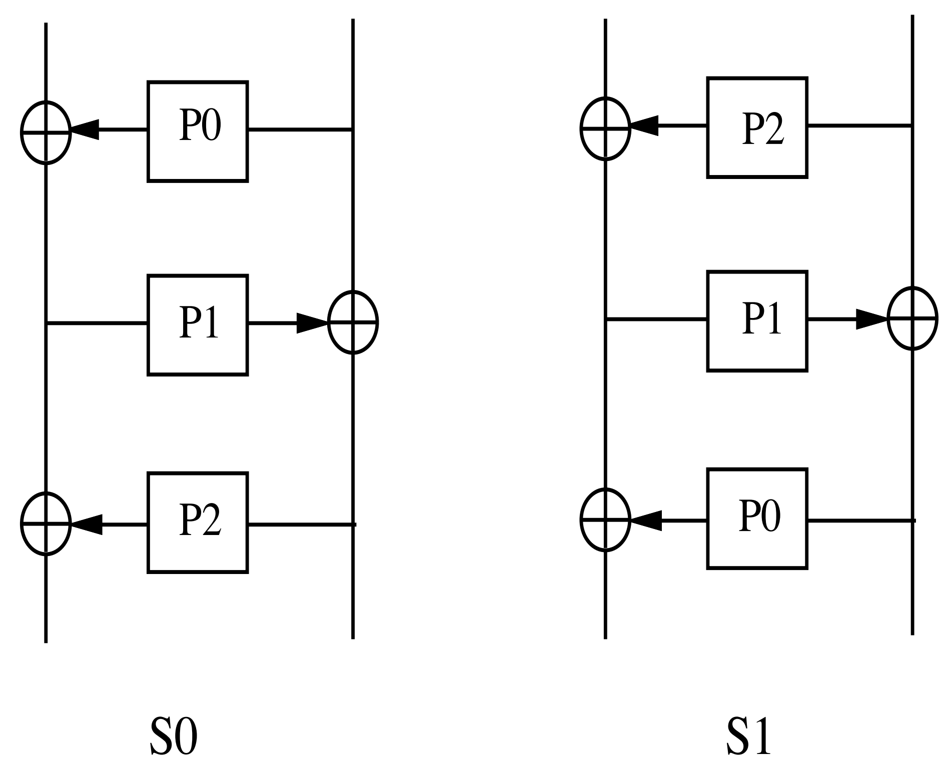
$$F(x, y) = (x + P_1(y) + P_3(y + P_2(x + P_1(y))), y + P_2(x + P_1(y)))$$

S-boxes constructed with 3-round Feistel structure

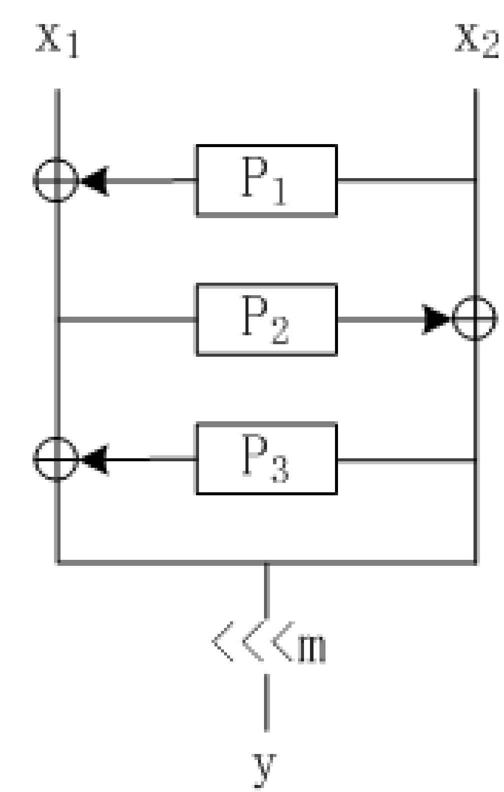
CS-CIPHER



CRYPTON



ZUC



Algorithm/S-box	Differential uniformity	Nonlinearity	Algebraic degree
CS-CIPER/P	16	96	5
CRYPTON/S_0, S_1	8	96	5
ZUC/S_0	8	96	5

Bounds on S-boxes constructed with 3-round Feistel structure

Bounds on differential uniformity

Let $F(x, y)$ be an S-box constructed as previous. Then

- If $P_2(x)$ is not a permutation over $\mathbb{F}_{2^k}$, then $\Delta(F) \geq 2^{k+1}$.
- If $P_2(x)$ is a permutation over $\mathbb{F}_{2^k}$, then $\Delta(F) \geq 2\Delta(P_2)$.

Bounds on S-boxes constructed with 3-round Feistel structure

Bounds on differential uniformity

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Bounds on nonlinearity

Let $F(x,y)$ be an S-box constructed as previous, $\lambda_k = \begin{cases} 2^{\frac{k+1}{2}} & k \text{ odd}, \\ 2^{\frac{k}{2}+1} & k \text{ even}. \end{cases}$.

If for any $a \in \mathbb{F}_{2^k}^*$, there exists $b \in \mathbb{F}_{2^k}^*$, such that $|\lambda_{P_2}(a,b)| \geq \lambda_k$, then
$\mathcal{NL}(F(x,y)) \leq \begin{cases} 2^{2k-1} - 2^k & k \text{ odd}, \\ 2^{2k-1} - 2^{k+1} & k \text{ even}. \end{cases}$

Bounds on S-boxes constructed with 3-round Feistel structure

Bounds of 8-bit S-boxes

Let $F_{P_1,P_2,P_3}(x,y)$ be an S-box over $\mathbb{F}_{2^4}^2$ constructed with 3-round Feistel structure. Then

- $\Delta(F_{P_1,P_2,P_3}) \geq 8$.
- If $\Delta(F_{P_1,P_2,P_3}) = 8$, then $\mathcal{NL}(F_{P_1,P_2,P_3}) \leq 96$.

Bounds on S-boxes constructed with 3-round Feistel structure

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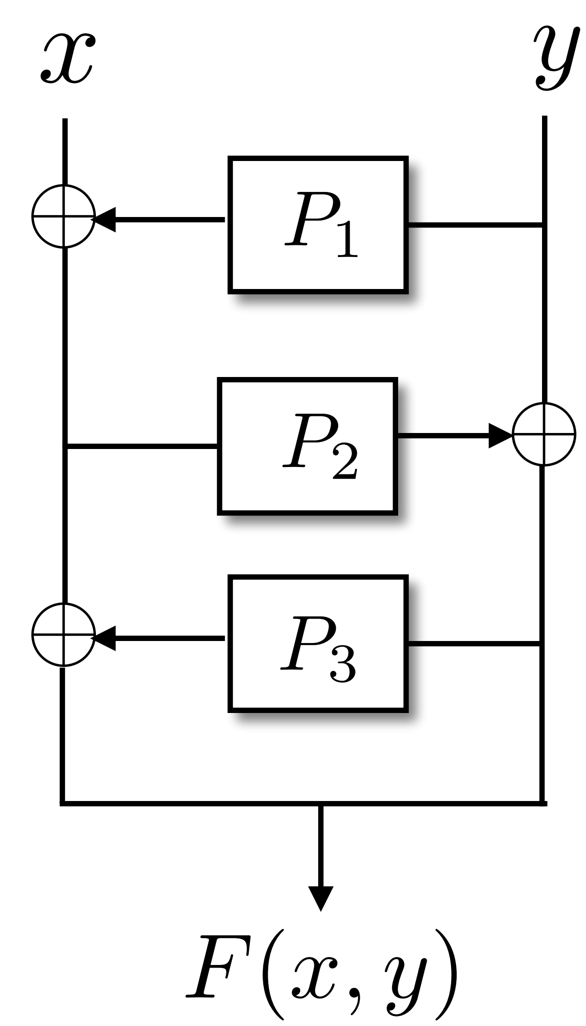
An improved example

$P_1 = x^3$, $P_2 = x + g^6 * x^{10} + g^3 * x^{13}$, $(g^4 + g + 1 = 0)$, $P_3 = \sum_{i=4}^{14} x^i$. F_{P_1,P_2,P_3}, F_{P_3,P_2,P_3} are with differential uniformity 8, nonlinearity 96, and algebraic degree 6.

General construction of big S-boxes

- n even, permutation; ✓
- Lowest differential uniformity; 4
- The best know nonlinearity; $2^{n-1} - 2^{\frac{n}{2}}$
- Easy implementation; ✓

P_2 must be an APN permutation



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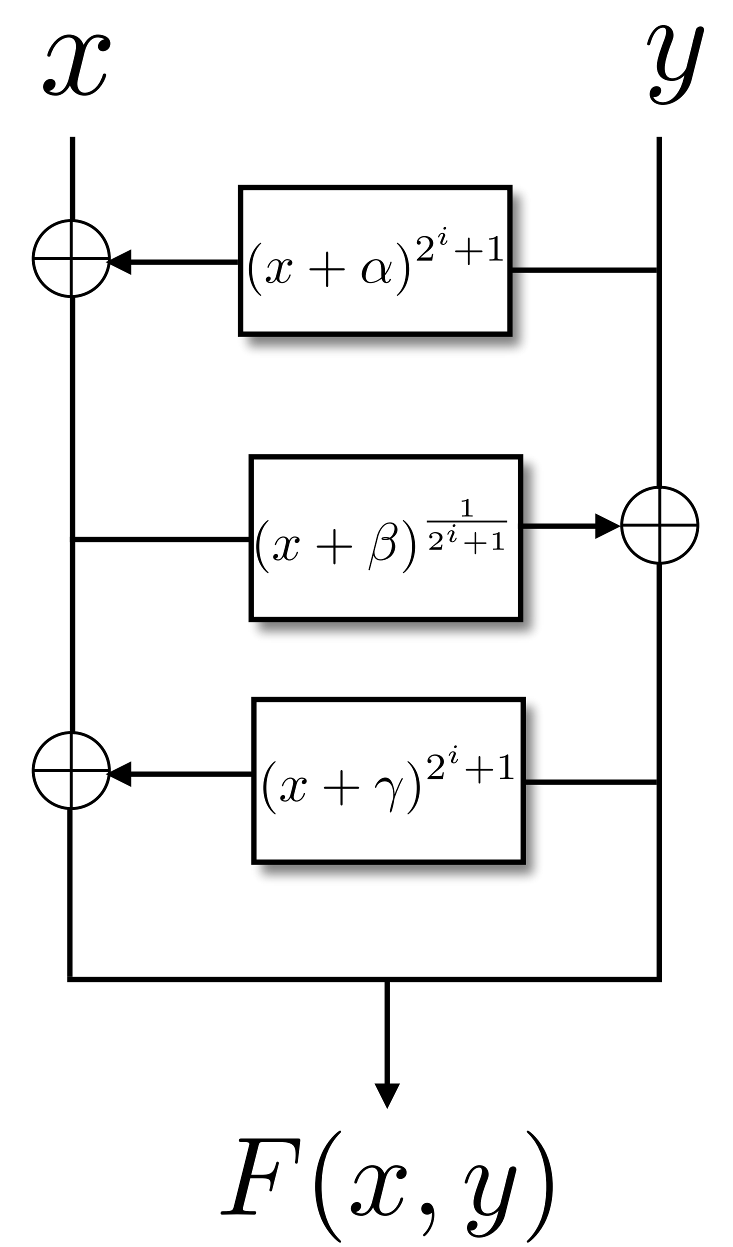
- Easy implementation; ✓

k odd, $(\lambda_k = 2^{\frac{k+1}{2}})$

- there are APN permutations over $\mathbb{F}_{2^k}$;
- $\mathcal{NL}(F) \leq 2^{2k-1} - 2^k$,

General construction of big S-boxes

- k odd, $\gcd(i,k) = 1$.
- x^{2^i+1} is an APN permutation.
- $x^{\frac{1}{2^i+1}}$ compositional inverse.
- $\alpha, \beta, \gamma \in \mathbb{F}_{2^k}$.



General construction of big S-boxes

Theorem

k odd, $\gcd(i,k)=1$. Let $F(x,y)=(x+(y+\alpha)^{2^i+1}+(y+\gamma+(x+\beta+(y+\alpha)^{2^i+1})^{\frac{1}{2^i+1}})^{2^i+1}, y+(x+\beta+(y+\alpha)^{2^i+1})^{\frac{1}{2^i+1}})$, be an S-box constructed as previous. Then

- When $\alpha=\gamma$, $F(x,y)$ is an involution on $\mathbb{F}_{2^k}^2$.
- Its differential uniformity equals 4. differential spectrum $\{0,4\}$.
- Its nonlinearity equals $2^{2k-1}-2^k$. Walsh spectrum $\{0,\pm 2^{k+1}\}$.
- Its algebraic degree equals k.

General construction of big S-boxes

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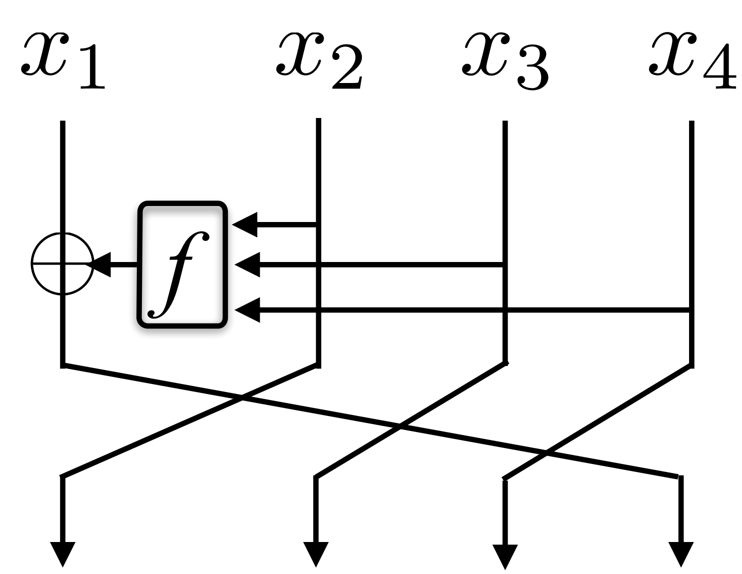
K. Aoki [SAC 98]: "Characterizing the F-functions whose maximum differential probability with keys is small"

Construction of S-boxes with unbalanced Feistel structure

$$x_i \in \mathbb{F}_{2^k},\ f : \mathbb{F}_{2^k}^3 \mapsto \mathbb{F}_{2^k}.\ \ P_f : \mathbb{F}_{2^k}^4 \mapsto \mathbb{F}_{2^k}^4$$

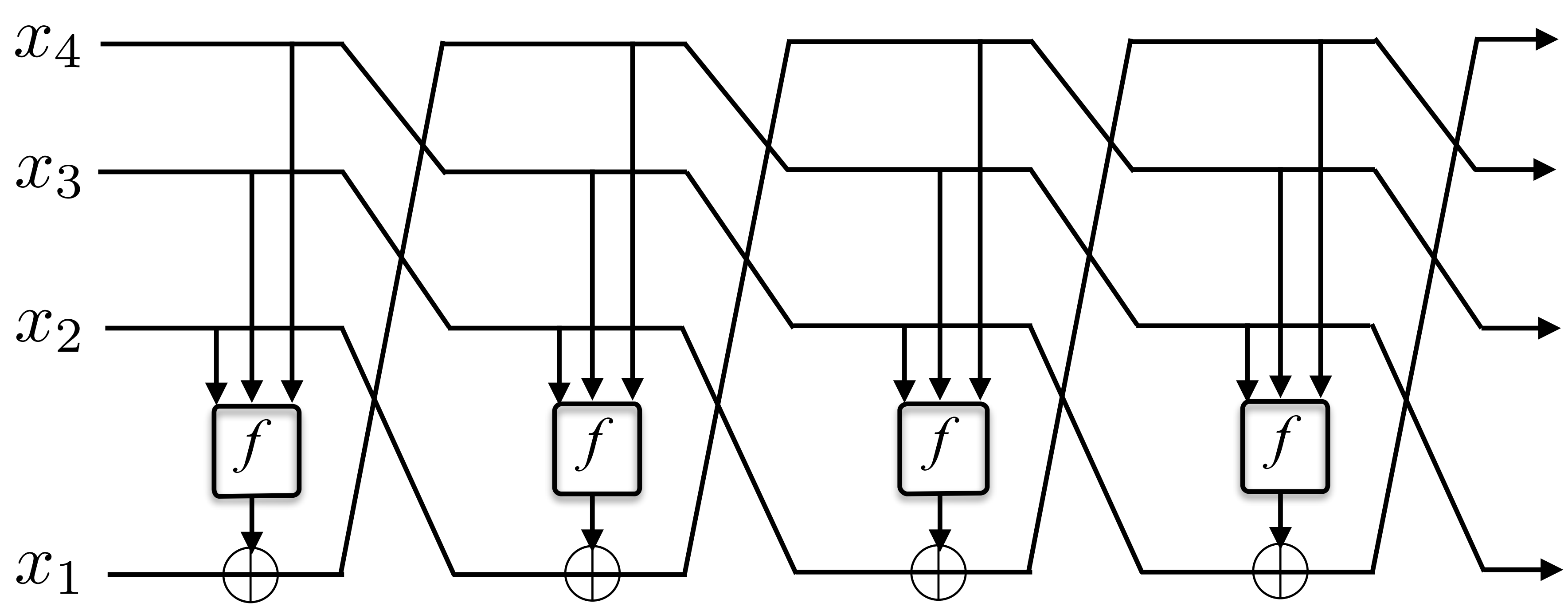


$$P_f(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4, x_1 + f(x_2, x_3, x_4))$$

$$P_f^t = P_f(P_f^{t-1}), P_f^1 = P_f$$

Construction of S-boxes with unbalanced Feistel structure

Implementation of P_f^4



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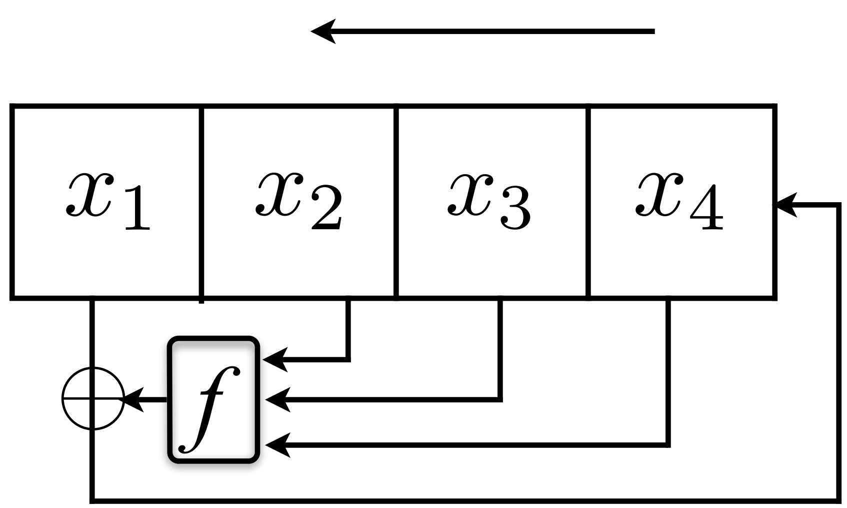


4 round NLFSR

Optimal 4-bit S-boxes

$$k = 1, x_i \in \mathbb{F}_2, P_f^4, P_f^5 : \mathbb{F}_2^4 \mapsto \mathbb{F}_2^4.$$

Optimal 4-bit S-box [G. Leander, A. Poschmann 07]

- A 4-bit S-box is called optimal if it is a permutation over $\mathbb{F}_{2^4}$ with differential uniformity 4 and nonlinearity 4.
- There are 16 classes of optimal 4-bit S-boxes up to affine equivalence.

Construction of optimal 4-bit S-boxes, 4-round

f	Operations	G_i	f	Operations	G_i
x_2x_3	(1, 1, 0)	8	x_2x_3+1	(1, 1, 1)	8
x_3x_4	(1, 1, 0)	8	x_3x_4+1	(1, 1, 1)	8
$(x_3+1)x_4$	(1, 1, 1)	8	$(x_3+1)x_4+1^\star$	(1, 1, 2)	8
$x_2(x_3+1)$	(1, 1, 1)	8	$x_2(x_3+1)+1^\star$	(1, 1, 2)	8
$x_3(x_4+1)$	(1, 1, 1)	8	$x_3(x_4+1)+1^\star$	(1, 1, 2)	8
$(x_2+1)x_3$	(1, 1, 1)	8	$(x_2+1)x_3+1^\star$	(1, 1, 2)	8
$(x_2+1)(x_3+1)+1$	(1, 1, 3)	8	$(x_2+1)(x_3+1)$	(1, 1, 2)	8
$(x_3+1)(x_4+1)+1$	(1, 1, 3)	8	$(x_3+1)(x_4+1)$	(1, 1, 2)	8
$x_2x_3+x_4$	(2, 1, 0)	8	$x_2x_3+x_4+1^\star$	(2, 1, 1)	8
$x_2+x_3x_4$	(2, 1, 0)	8	$x_2+x_3x_4+1^\star$	(2, 1, 1)	8
$x_2+(x_3+1)x_4$	(2, 1, 1)	8	$x_2+(x_3+1)x_4+1$	(2, 1, 2)	8
$(x_2+1)x_3+x_4$	(2, 1, 1)	8	$(x_2+1)x_3+x_4+1$	(2, 1, 2)	8
$x_2+x_3(x_4+1)$	(2, 1, 1)	8	$x_2+x_3(x_4+1)+1$	(2, 1, 2)	8
$x_2(x_3+1)+x_4$	(2, 1, 1)	8	$x_2(x_3+1)+x_4+1$	(2, 1, 2)	8
$x_2+(x_3+1)(x_4+1)+1$	(2, 1, 3)	8	$x_2+(x_3+1)(x_4+1)^\star$	(2, 1, 2)	8
$(x_2+1)(x_3+1)+x_4+1$	(2, 1, 3)	8	$(x_2+1)(x_3+1)+x_4^\star$	(2, 1, 2)	8
$x_2(x_3+x_4)+x_3x_4$	(3, 2, 0)	1	$x_2(x_3+x_4)+x_3x_4+1$	(3, 2, 1)	1
$x_2(x_4+x_3+1)+(x_3+1)x_4$	(3, 2, 1)	1	$x_2(x_4+x_3+1)+(x_3+1)x_4+1$	(3, 2, 2)	1
$x_2(x_3+x_4+1)+x_3(x_4+1)$	(3, 2, 1)	1	$x_2(x_3+x_4+1)+x_3(x_4+1)+1$	(3, 2, 2)	1
$(x_2+1+x_4)x_3+(x_2+1)x_4$	(3, 2, 1)	1	$(x_2+1+x_4)x_3+(x_2+1)x_4+1$	(3, 2, 2)	1

Table 2. Boolean functions such that P_f^4 are optimal 4-bit S-boxes

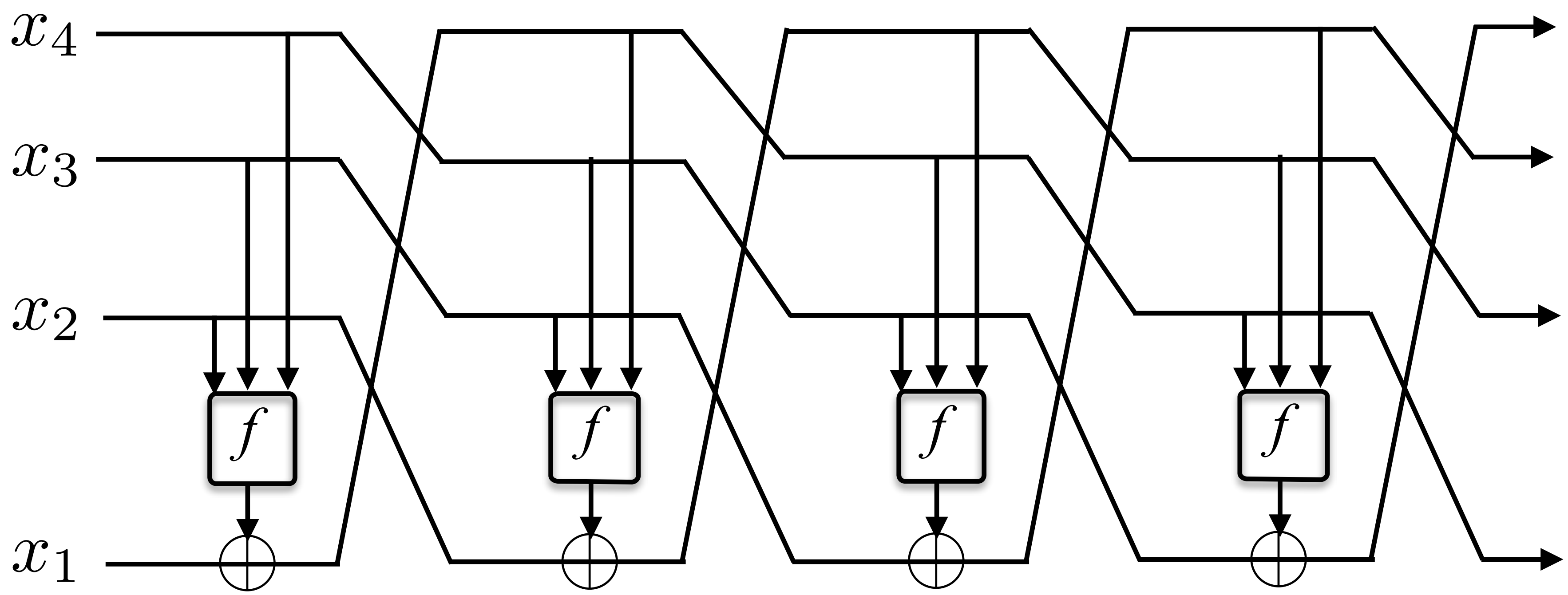
Construction of optimal 4-bit S-boxes, 5-round

f	Operations	G_i	f	Operations	G_i
$x_2(x_3+x_4)+1$	(2, 1, 1)	7	$(x_2+x_4)x_3+1^\star$	(2, 1, 1)	4
$(x_2+x_3)x_4+1$	(2, 1, 1)	7	$(x_2+x_4)(x_3+1)+1^\star$	(2, 1, 2)	4
$(x_2+x_3)(x_4+1)+1$	(2, 1, 2)	7	$(x_2+1)(x_3+x_4)+1$	(2, 1, 2)	7
$x_2x_3+(x_2+1)x_4$	(2, 2, 1)	13	$x_2(x_4+1)+x_3x_4$	(2, 2, 1)	13
$x_2x_4+x_3(x_4+1)+1$	(2, 2, 2)	13	$x_2(x_3+1)+x_3(x_4+1)$	(2, 2, 2)	4
$(x_2+1)x_3+x_2x_4+1$	(2, 2, 2)	13	$x_2x_4+(x_3+1)(x_4+1)^\star$	(2, 2, 2)	13
$x_2x_3+(x_2+1)(x_4+1)^\star$	(2, 2, 2)	13	$(x_2+1)(x_4+1)+x_3x_4^\star$	(2, 2, 2)	13
$(x_2+1)(x_3+1)+x_2x_4^\star$	(2, 2, 2)	13	$(x_2+1)x_3+(x_3+1)x_4$	(2, 2, 2)	4
$x_2((x_3+1)x_4+1)+x_3(x_4+1)$	(2, 3, 3)	11	$(x_2(x_4+1)+1)x_3+(x_2+1)x_4$	(2, 3, 3)	11
$(x_2x_3+1)x_4+(x_2+1)(x_3+1)$	(2, 3, 3)	11	$x_2(x_3x_4+1)+(x_3+1)(x_4+1)$	(2, 3, 3)	11
$(x_2x_3+1)x_4+(x_2+1)(x_3+1)+1$	(2, 3, 4)	11	$(x_2x_4+1)x_3+(x_2+1)(x_4+1)+1$	(2, 3, 4)	3
$x_2(x_3x_4+1)+(x_3+1)(x_4+1)+1$	(2, 3, 4)	11	$x_2(x_3(x_4+1)+1)+(x_3+1)x_4+1$	(2, 3, 4)	3
$(x_2(x_4+1)+1)x_3+(x_2+1)x_4+1$	(2, 3, 4)	11	$x_2((x_3+1)x_4+1)+x_3(x_4+1)+1$	(2, 3, 4)	11

Table 3. Boolean functions such that P_f^5 are optimal 4-bit S-boxes

Construction of 8-bit S-box with unbalanced Feistel structure

$$k = 2, x_i \in \mathbb{F}_{2^2}, P_f^4 : \mathbb{F}_{2^2}^4 \mapsto \mathbb{F}_{2^2}^4.$$



For any f in Table 2, P_f^4 is an 8-bit S-boxes with

- differential uniformity 16;
- nonlinearity 96.

THANK YOU!

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